Homework - S403

Due September 17, 2013

Please write on only one side of the sheet.

1. Let G be a group.

(a) For each $g \in G$ the centralizer of g is the set $C(g) = \{x \in G | gx = xg\}$. Prove that $C(g)$ is a subgroup of G. Compute the centralizers of H and R_2 in D_4.

(b) The center of G is the set $Z(G) = \{g \in G | gx = xg \text{ for all } x \in G\}$. Prove that $Z(G)$ is a subgroup. Compute $Z(D_4)$.

2. Prove that if G is a group of even order, then G contains an element of order 2.

3. Let G be a group and let $g \in G$ have finite order m.

(a) Prove that if $n|m$, then $o(g^n) = g^{m/n}$.

(b) Prove that if k is an integer and $d = gcd\{m, k\}$, then $o(g^k) = m/d$.

(c) Prove that if G is a finite group of order p^r where p is a prime, then G contains an element of order p.

4. (a) Let G be an abelian group and let $x, y \in G$. Let $o(x) = m$ and $o(y) = n$. Prove that if m and n are relatively prime then $o(xy) = mn$.

(b) Prove that the statement of part (a) is false for arbitrary groups. (Hint: Consider S_3.)

(c) Let G be an abelian group of order 6. Prove G is cyclic.

5. Let G be a finite group and let H be a subgroup. For each $g \in G$ define the H-order of g to be the smallest positive integer m such that $g^m \in H$. (Note that there is *some* positive integer m such that $g^m \in H$ because there is a positive integer m such that $g^m = e$ and $e \in H$.) Prove that if m is the H-order of g then for all integers k, $g^k \in H$ if and only if $m|k$.

6. Let H be a subgroup of a group G.

(a) Prove that if $g \in G$, then gHg^{-1} is a subgroup of G.

(b) Prove that if $g_1, g_2 \in G$ and $g_1 H = g_2 H$, then $g_1 H g_1^{-1} = g_2 H g_2^{-1}$.

Now assume G is finite.

(c) Prove that if $g \in G$, then $|gHg^{-1}| = |H|$.

(d) Prove that if $G = \cup_{g \in G} gHg^{-1}$, then $H = G$.